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ON THE BOUNDARY CONDITION ON THE EARTH SURFACE FOR DIFFUSING POLLUTION

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The diffusion of pollution along the vertical will be described here by means of the routine parabolic equation

$$\frac{\partial q}{\partial t} - w\frac{\partial q}{\partial z} = \frac{\partial}{\partial z}K(z)\frac{\partial q}{\partial z} \tag{1}$$

where q is the concentration of the pollution, t the time, w the velocity of gravitational fall of pollution particles, and $K(z)$ the coefficient of vertical diffusion. The vertical pollution flux is then $K(z)\frac{\partial q}{\partial z} + wq$. The boundary condition on the earth surface must reflect the character of the interaction between the diffusing particles and the surface; such a condition can be written as

$$\left(K\frac{\partial q}{\partial z} + wq - \beta q\right)_{z=z_0} = 0 \tag{2}$$

where β is a constant characterizing the interaction between the diffusing pollution and the surface. When $\beta = 0$ the condition (2) means that the pollution flux through the earth surface is zero, so that the entire pollution remains in the air; in other words, the pollution particles are reflected from the earth surface. When $\beta = \infty$ the condition (2) takes a form of $(q)_{z=z_0} = 0$; it means that the pollution particles reaching the earth surface stick to the earth or are absorbed by it.

When $0 < \beta < \infty$ the condition (2) means that the particles reaching the earth surface can be either reflected from or absorbed by it. In order to find the order of magnitude of β, let us consider the simplest case $K(z) = \text{const}$, $z_0 = 0$. The solution of Equation (1) at the height $z = h$ and at the moment when there is an instantaneous point source of unit intensity $t = 0$ is, under the boundary condition (2),

$$q(z,t;h) = \frac{1}{2\sqrt{(\pi K t)}}e^{-(w(z-h)/2K)-(w^2t/4K)}\left[e^{-((z-h)^2/4Kt)} + e^{-((z+h)^2/4Kt)}\right] - \frac{K}{2\beta - w}e^{(\beta h/K)+((\beta-w)z/K)+(\beta(\beta-w)t/K)}\phi\left[-\frac{z+h+(2\beta-w)t}{\sqrt{(2Kt)}}\right] \tag{3}$$

where $\phi(x)$ is the standardized normal integral probability distribution. The fraction of the diffusing particles to be absorbed by the earth surface at the moment t is determined by the expression

$$(4)\qquad P(t;h) = 1-\int_0^\infty q(z,t;h)\,dz = \frac{\beta}{\beta-w}e^{wh/K}\phi\left(-\frac{h+wt}{\sqrt{(2Kt)}}\right) + \phi\left(-\frac{h-wt}{\sqrt{(2Kt)}}\right) - \frac{2\beta-w}{\beta-w}e^{(\beta h/K)+(\beta(\beta-w)t/K)} \times \phi\left[-\frac{h+(2\beta-w)t}{\sqrt{(2Kt)}}\right].$$

The value $P(t;h)$ is the probability of the event that the particle, being at the moment $t=0$ in the point $z=h$, will be absorbed during the time t by the earth surface. In particular, the particle reaching the earth surface at the moment $t=0$, will be absorbed by it during the time t with the probability

$$(5)\quad P(t;0) = 1+\frac{w}{\beta-w}\phi\left(-w\sqrt{\frac{t}{2K}}\right)-\frac{2\beta-w}{\beta-w}e^{(\beta(\beta-w)t/K)}\phi\left[(w-2\beta)\sqrt{\frac{t}{2K}}\right]$$

When t is small this expression takes the form

$$(6)\qquad P(t;0) = 2\beta\sqrt{\left(\frac{t}{\pi K}\right)}+0(\sqrt{t})$$

so that the value β can be determined by

$$(7)\qquad \beta = \lim_{t\to 0}\frac{\sqrt{(\pi K)}\,P(t;0)}{2\sqrt{t}}$$

This formula throws light on the statistical value of the parameter β. Note that if the process of the absorption of the particles obeyed the Poisson law, then $P(t,0)$ if t is small would be proportional to t and not to $\sqrt{t}$. Consequently, the boundary condition (2) implies that the absorption of particles reaching the earth surface occurs more often than in the case of the Poisson law.

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DESCRIPTION OF TURBULENCE IN TERMS OF LAGRANGIAN VARIABLES

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The method of describing turbulence in terms of Lagrangian variables is in giving statistical characteristics of the state of an "indicator" system (an indicator is a selected particle of a flux, called also "a floating guard"). The state of each indicator is determined by its coordinate x and velocity v.[1] A great amount of information is required to give a description of the instantaneous state of a hydrodynamical field in detail. But the consideration of the statistical characteristics of the track of a single indicator may present valuable information on the properties of the flux in the form of a distribution function $\varphi(x, v)$. In particular, it makes it possible to obtain all one-point characteristics of turbulence (as the mean flux velocity at any point, the intensity of pulsation, Reynolds stress, and so on) and some characteristics of turbulent diffusion also. The description of the flux with two indicators is similar and leads to the consideration of a double-distribution function and of two-point characteristics of turbulence related with it (correlation moments).

The conditional distribution function $\varphi_\tau(x, v)$ is the principal characteristic of the motion in Lagrangian variables, i.e. the probability of the selected particle, which at the initial moment has the coordinate x_0 and the velocity v_0, having after a certain period of time τ the coordinate x and the velocity v. The problem is to determine this function theoretically for a "free turbulence" (i.e. not taking into consideration the boundary influence and mechanism of energy supply) when we assume that:

(*a*) The evolution of the state of the selected particle in time forms a Markov process and can be described by the Foker-Plank equation in the space (x, v).

(*b*) The equations describing the process of particle floating in the turbulent flux are invariant with respect to Gallileo's transformations including translations and rotations. The situation is similar to that in

[1] v is a macroscopic flux velocity for a definite place which is, generally speaking, different from $\dot{x}$ (if the diffusion motion of the indicator itself is taken into account).

the Kolmogoroff theory for the Eulerian description of local isotropic turbulence.

Under such conditions the distribution function φ obeys the equation:

$$\frac{\partial \varphi}{\partial t} = -(\vec{v}\nabla)\varphi + B\Delta_v \varphi$$

where Δ_v is the Laplace operator in the space of velocities, B is the main characteristic of the process and is expressed in cm²/sec³.

FIG. 1. Virtual diffusion coefficient K (cm²/sec.) and root mean square pulsation velocity v' (cm/sec.) as functions of the scale of turbulence l.

The solution of this equation under the initial condition

$$\varphi_0(x, v) = \delta(x - x_0)\,\delta(v - v_0)$$

gives the function $\varphi_\tau(x, v)$ and allows one to obtain explicit expressions for $\overline{\Delta_\tau v^2}$, $\overline{\delta x^2} = \overline{(\Delta x - v_0 \Delta t)^2}$ and also the diffusion coefficient K determined by the formula:

$$K = \frac{1}{2}\frac{d}{dt}(\overline{\delta x^2})$$

It appears that $\overline{\delta x^2} \sim B\tau^3$; $K \sim B\tau^2$; $\overline{\Delta v^2} \sim B\tau$. If the characteristic scale l for the probability distribution is introduced (putting, for example, $l^2 = \overline{\delta x^2}$) and the interval of time τ is eliminated from the above equations then we obtain:

$$\overline{\Delta v^2} \sim (Bl)^{2/3}; \quad K = B^{1/3} l^{4/3}.$$

These results are in good agreement with the Kolmogoroff theory if the coefficient B is interpreted as the energy dissipation ϵ (with an accuracy of a multiplier of the order of unity). The agreement of the results with the conclusions obtained solely from the similarity hypothesis, is an argument in favour of hypothesis (*a*).

In a "free turbulence", as is known, a stationary distribution of probabilities for the velocities themselves does not exist (the distribution is supposed to exist only for corresponding differences in velocities) and consequently, the turbulent coefficient of diffusion increases indefinitely with time (cloud size). By formally introducing a weak friction force into the equation, taken to be linearly dependent on the flux velocity, we obtain a scheme with a final relaxation time T. When $B = 1$ cm² sec³ and $T = 5 \cdot 10^5$ sec. (six days) this scheme (Fig. 1) allows us to describe the variation of the horizontal coefficient of turbulent diffusion to an upper limit of 10^{11} cm²/sec. and the average level of turbulent energy in the atmosphere also.

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SMOKE PROPAGATION IN THE SURFACE LAYER OF THE ATMOSPHERE

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1. The Principle of a Limited Velocity of Atmospheric Diffusion

The diffusion of pollution (in particular, smoke) in the atmosphere is due to the turbulent pulsations of the wind velocity. The magnitude of these pulsations is limited (for example, it does not exceed the sound velocity): therefore the following principle may be set forth:

(a) *Propagation of the pollution through space due to Atmospheric diffusion occurs with a limited velocity.*

In accordance with this principle, the space occupied with a smoke flowing out of any source, has a very distinct boundary beyond which there is no smoke; such a boundary can visually be seen while observing the diffusing smoke.

Let us consider a single puff of smoke. If the maximum velocity of the vertical propagation of the diffusing smoke is designated as w^*, the change of the vertical diameter of the smoke puff with time will be described with a formula:

$$D = 2w^* t. \tag{1}$$

This formula was verified experimentally by Kasanski and Monin (1957) by means of filming a single smoke-puff in the surface layer (with frequency of 1 frame per 5 sec.). An example of such a film is given in Fig. 1. The dependence of D upon t is shown in Fig. 2(a) and is in a good agreement with Equation (1) (w^* in this case has the value $0 \cdot 12$ m/sec.).

Fig. 2(b) displays the dependence of the horizontal diameter of the smoke puff D_1 upon t: the indicated dependence can be obtained theoretically from the equation

$$\frac{dD_1}{dt} = 2u^* + \Delta_D \bar{u} \tag{2}$$

where u^* is the maximum velocity of the horizontal diffusion and $\Delta_D \bar{u}$ is the difference between the wind velocity on the upper and lower boundaries of the smoke puff. $\Delta_p \Xi$ increases with D, and consequently D_1 increases faster than the linear function of t.

The method widely used for describing the concentration in a smoke puff, is the Gaussian function. It is stated then that the visible boundary of the smoke puff corresponds to some critical concentration at which the air becomes opaque. We disagree with this interpretation and consider that the concentration beyond the visible smoke puff equals zero.

2. The Use of the Similarity Theory for Describing the Turbulent Diffusion in the Surface Layer

Atmospheric diffusion is characterized by the state of turbulence. The stationary turbulent regime in the surface layer, when the turbulence is homogeneous in the horizontal, obeys the similarity theory developed by Monin and Obukhov (1953, 1954). The turbulent regime is completely determined by the parameters $v_* = (\tau/\rho)^{1/2}$ and $q/c_p\rho$ (v_* is the friction velocity, τ the turbulent stress, q the turbulent heat flux, ρ the air density, c_p the specific heat), which do not vary with altitude in the surface layer, and by the universal parameter g/T_0 (g is the acceleration of gravity, T_0 the mean temperature of the surface layer) characterizing the effect of the Archimedes forces. According to the similarity theory, the only scale of velocity in the surface layer is v_* and the only scale of length is the value

$$L = \frac{v_*^3}{\kappa \frac{g}{T_0}\left(-\frac{q}{c_p\rho}\right)} \tag{3}$$

(κ = von Kármán constant introduced for convenience). Under indifferent stratification $q = 0$ and $|L| = \infty$; under stable stratification $q < 0$ and $L > 0$, and under convection $q > 0$ and $L < 0$. All dimensionless variables can be functions only of the dimensionless height $\zeta = \frac{z}{L}$ (z is the height). In particular, the averaged wind velocity can be represented by

$$\bar{u}(z) = \frac{v_*}{\kappa}\left[f\left(\frac{z}{L}\right) - f\left(\frac{z_0}{L}\right)\right] \tag{4}$$

where z_0 is the roughness, and $f(\zeta)$ is the universal function. As stated in Monin and Obukhov (1953, 1954), if $|\zeta|$ is small, $f(\zeta)$ has the following form

$$f(\zeta) \approx \ln|\zeta| + \beta\zeta \tag{5}$$

where $\beta \approx 0 \cdot 6$. The parameter ζ is connected with the Richardson number by the relation

$$\frac{Ri}{Ri_{Kp}} = \frac{1}{f'(\zeta)}, \tag{6}$$

To describe the turbulent diffusion in the surface layer of the atmosphere the following principle can be formulated:

(*b*) *The turbulent diffusion in a horizontally-homogeneous stationary surface layer of air obeys the similarity theory in which the values L and v_* are the only scales of length and velocity.*

According to this principle the maximum velocity of the vertical propagation of the diffusing smoke is given by

$$w^* = \lambda v_* \varphi(\zeta) \tag{7}$$

where $\varphi(\zeta)$ is a certain universal function, which can be subjected to the condition $\varphi(0) = 1$ so that λ will be equal to w^*/v^* under indifferent stratification. In the experiments shown in Fig. 2 which were carried out under conditions close to indifferent stratification $w^* = 0 \cdot 12$ m/sec. and $v^* = 0 \cdot 16$ m/sec. so that $\lambda = 0 \cdot 75$. Perepelkina's (1957) data show that $(\overline{w'^2})^{1/2}$ on average equals $0 \cdot 86\ v_*$ (w' is the vertical components of the wind velocity).

In order to determine the form of the function $\varphi(\zeta)$, let us use the turbulent energy balance equation

$$K\left(\frac{d\bar{u}}{dz}\right)^2 - \frac{g}{T_0}\alpha K\frac{d\overline{T}}{dz} = \epsilon \tag{8}$$

where K is the coefficient of turbulence, ϵ is the rate of dissipation of the turbulent energy, α is the ratio of the diffusion coefficients for momentum and heat. Rewriting this equation in the form

$$\frac{\epsilon}{K\left(\frac{d\bar{u}}{dz}\right)^2} = 1 - \alpha Ri, \tag{9}$$

using the equation $K\frac{d\bar{u}}{dz} = v_*^2$ and putting, in accordance with the similarity theory, $\epsilon \propto \frac{w^{*3}}{l}$ and $K \propto w^* l$ where l is the scale of turbulence, we get

$$\left(\frac{w^*}{v^*}\right)^4 \propto 1 - \alpha Ri \tag{10}$$

from which

$$\varphi(\zeta) = (1-\alpha Ri)^{1/4} = \left[1 - \frac{1}{f'(\zeta)}\right]^{1/4} \tag{11}$$

as α has, obviously, the value $1/Ri_{Kp}$. When ζ is small and the approximation (5) is valid we get

$$\varphi(\zeta) \approx 1 - \frac{\zeta}{4}. \tag{12}$$

3. The Shape of the Boundaries of the Smoke Plume

Let us consider a smoke plume of neutral temperature flowing out of a stationary point source at the height h in the surface layer of the atmosphere. The equations of motion of smoke particles at the upper boundary of the plume have the form

$$\frac{dx}{dt} = \bar{u}; \qquad \frac{dz}{dt} = w^{*} \tag{13}$$

where x is the horizontal coordinate in the wind direction. Using Equations (4), (7) and (11), we get from (12) the following differential equation for the upper boundary of the smoke plume

$$\frac{dz}{dx} = \frac{1}{\kappa\lambda}\,\frac{f\left(\frac{z}{L}\right) - f\left(\frac{z_0}{L}\right)}{\left[1 - \frac{1}{f'(z/L)}\right]^{1/4}}. \tag{14}$$

A similar equation (with a minus sign) is obtained for the lower boundary of the smoke plume. The equation does not contain explicitly the friction velocity, but does contain the stratification parameter z/L. Hence the following conclusion can be drawn:

(c) *The shape of the boundaries of the smoke plume (in particular, their inclination to the horizon) does not depend upon the wind velocity, but does depend upon the stratification of atmosphere.*

When $|z/L|$ is small and the approximation (5) is valid, the Equation (14) can be written in the form:

$$\operatorname{tg}\alpha = \frac{dz}{dx} \approx \kappa\lambda\,\frac{1 - \frac{\zeta}{4}}{\ln\frac{z}{z_0} + \beta\zeta}. \tag{15}$$

Putting $\kappa = 0\cdot 4, \lambda = 0\cdot 8, z/z_0 = 800$ in the case of indifferent stratification we get $\operatorname{tg}\alpha = 0\cdot 05$.

By integrating Equation (14) we get

$$x = \frac{L}{\kappa\lambda}\left[F\left(\frac{z}{L}, \frac{z_0}{L}\right) - F\left(\frac{h}{L}, \frac{z_0}{L}\right)\right]; \quad F(\zeta, \zeta_0) = \int_{\zeta_0}^{\zeta} \frac{f(\zeta) - f(\zeta_0)}{\left[1 - \frac{1}{f'(\zeta)}\right]^{1/4}}\, d\zeta. \tag{16}$$

This result is one of those few in diffusion theory where one can take into account the variation of the wind velocity with altitude. Taking $f(\zeta)$ from the empirical graph of this function published by Monin and Obukhov (1953, 1954), and by numerical integration of (16) we have plotted the graph for the shape of the upper boundary of the smoke plume $\zeta = F^{-1}(\xi, \zeta_0)$, where $\xi = \frac{\kappa\lambda x}{L}$. These graphs are given in Fig. 3 (the solid curve is for the stable stratification; the dashed curve is for the unstable stratification) and it is seen that the smoke plume grows vertically with distance from the source much faster under unstable than under stable stratification.

The calculation given above applies to a smoke of neutral temperature. But it often happens that one has to deal with a heated smoke. On flowing up to a certain height h a heated smoke acquires the temperature of the surrounding medium and then diffuses as a neutral plume. The heated plume far from the source becomes similar to the smoke plume with a neutral temperature flowing out of a source elevated to the height h. The height h must depend upon the intensity of turbulent mixing, which can be characterized by the value v_*. Some authors have therefore sought to express the dependence of h mainly upon the wind velocity $\bar{u}$.

The conclusions mentioned above were verified by Kasanski and Monin (1957) with the help of surface sources. We tried to imitate the steady linear source of smoke perpendicular to the wind direction. The smoke plume was filmed from one side (with frequency of 1 frame per 15 sec.) and by combining the frames we obtained the average curve for the plume boundary; this curve was approximated by a straight line inclined to the horizon and the effective height h reached by the smoke, because of its heating, was determined. Fig. 4 shows examples of the smoke plumes and their treatment under unstable stratification and Fig. 5 under indifferent stratification. The cases with an inversion layer at a small height having a typical break in temperature profile were not treated (such cases can be observed during the formation and destruction of surface inversions, i.e. under non-stationary conditions).

The shape of the boundary of the smoke plume is in good agreement with the theory set forth if λ is close to or a little exceeds unity (the values of λ determined by such a method characterize the turbulence of scale large compared with the thickness of the smoke plume). The experiments completely confirmed the conclusion (c). Fig. 6 displays the measured dependence of the inclination of the plume boundary $\mathrm{tg}\,\alpha$ upon the stratification parameter $1/L$. Fig. 7 displays the dependence of the height h, to which the smoke ascends, upon the value v_*.

4. Concentration Profiles in a Smoke Plume

Distribution of the smoke concentration in space for one or another source can be determined theoretically only as the solution of the diffusion equation. Guided by the principle (a) we are denied the use of the routine parabolic diffusion equation corresponding to an infinitely rapid pollution propagation in space. The diffusion equation corresponding to the limited propagation velocity should be hyperbolic: such a hyperbolic system of equations was obtained by Monin (1955, 1956) in the form

$$\frac{\partial s}{\partial t}+\frac{\partial S}{\partial z}=0; \quad \frac{\partial S}{\partial t}+2aS=-w^*\frac{\partial w^* s}{\partial z} \tag{17}$$

where s is the concentration, and S is the turbulent flux of the diffusing pollution, a is a typical frequency of turbulent pulsations which in accordance with the similarity principle can be written in the form:

$$a=\frac{v_*}{z}\psi\left(\frac{z}{L}\right). \tag{18}$$

Using the formulae (7) and (11) for w^* and knowing that the stationary solution of Equation (17) has the form

$$s(z_2)-s(z_1)=-\frac{S}{\kappa v_*}\left[f\left(\frac{z_2}{L}\right)-f\left(\frac{z_1}{L}\right)\right] \tag{19}$$

which results from the similarity theory of Monin and Obukhov, we have to put

$$\psi(\zeta)=\frac{\lambda^2}{2\kappa}\zeta f'(\zeta)\left[1-\frac{1}{f'(\zeta)}\right]^{1/2}. \tag{20}$$

(In the process of obtaining this equation the expression $w^*\frac{\partial w^* s}{\partial z}$ was approximated by $w^{*2}\frac{\partial s}{\partial z}$, i.e. the pollution flux in zero concentration gradient, arising from the variation of turbulent intensity with height, was neglected.)

Equation (17) with the coefficients of (7)–(11) and (20) can only be solved numerically. However, in the case of indifferent stratification when $w^*=\lambda v^*$ and $a=\frac{\lambda^2}{2\kappa}\frac{v_*}{z}$ it becomes easy to find the solution of these equations, corresponding to the instantaneous point source of intensity Q. This solution has the form

$$s(z,t)=\frac{Q}{\kappa v_* t}\frac{\left(1-\frac{z}{\lambda v_* t}\right)^{\epsilon-1}}{\left(1+\frac{z}{\lambda v_* t}\right)^{\epsilon+1}}; \quad 0\leqslant z\leqslant \lambda v_* t \tag{21}$$

where $\epsilon=\frac{\lambda}{2\kappa}\geqslant 1$. If one has a stationary linear source of pollution, perpendicular to the wind direction, and one neglects the horizontal mixing and the change of wind velocity with height, the concentration of the pollution is given by:

$$s(x, z) = \frac{Q}{\kappa v_* x} \frac{\left(1 - \frac{\bar{u}z}{\lambda v_* x}\right)^{\epsilon - 1}}{\left(1 + \frac{\bar{u}z}{\lambda v_* x}\right)^{\epsilon + 1}}; \quad 0 \leq z \leq \frac{\lambda v_* x}{\bar{u}}. \tag{22}$$

Consequently, the concentration profiles at different distances from the sources are similar, and the maximum (surface) concentration is inversely proportional to the distance from the source. It is of interest to find out whether these results are at least approximately true also in the general case; i.e. under any stratification and taking into account the change of wind velocity with altitude. We shall formulate the hypothesis that:

$$s(x, z) = s_m(x) \phi\left[\frac{z}{H(x)}\right] \tag{23}$$

where $s_m(x)$ is the maximum concentration, and $H(x)$ is the height of the smoke plume at the distance x from the source. It follows from (23) that, approximately, $s_m \propto 1/x$. Indeed, under the condition of constancy of the total smoke flux through the plane $x = \text{const}$, i.e.

$$\int_{z_0}^{H} s\bar{u}\, dz = \text{const}, \tag{24}$$

we obtain $s_m \propto 1/\bar{u}_{cp} H$, where $\bar{u}_{cp}$ is the average value of the wind velocity in the layer from z_0 to H weighted by the function $\phi(z/H)$. Putting $H \approx x \operatorname{tg} \alpha$ and taking into consideration that $\operatorname{tg} \alpha$ and $\bar{u}_{cp}$ vary with the distance very slowly, we obtain $s_m \propto 1/x$.

The hypothesis (23) was also verified by Kasanski and Monin (1957) by data on smoke concentration from surface sources obtained by means of a sampling method. Fig. 8 displays the results of measurement of the function $\phi(z/H) = s/s_m$ from the data of 20 experiments. The dashed lines show the values of $\phi(\zeta) = (1-\zeta)^{\epsilon-1}(1+\zeta)^{-\epsilon-1}$ corresponding to the formula (22) when $\epsilon = 1$ and $\epsilon = 5/4$. The graph shows that the experimental points generally agree satisfactorily with the theoretical curve if $\epsilon = 5/4$. Large scatter for small z/H is due to the heat of source producing different lifts in different experiments and so to variations in the height of maximum concentration. Fig. 9 illustrates the results of measurements of the smoke concentration at a height of $1 \cdot 5$ m. at different distances $\xi = x/L$ from the source. The ordinate is $\lg(s(\xi)/s(\xi_0))$ where $\lg \xi_0 = 0 \cdot 2$. The graph shows that, approximately, $s(\xi) \propto 1/\xi$.

So the hypothesis (23) satisfactorily agrees with experimental data and we can draw the following conclusion:

(*d*) *The concentration profiles in a smoke plume at different distances from the source are approximately similar to each other. The maximum concentration in the smoke plume is approximately inversely proportional to the distance from the source.*

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TURBULENT DIFFUSION IN THE SURFACE LAYER UNDER STABLE STRATIFICATION

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Turbulent diffusion along the vertical will be described by means of the equation

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial z}\left\{k(z)\frac{\partial c}{\partial z}\right\} \tag{1}$$

where c is the concentration of the pollution, t the time, z the vertical coordinate, $k(z)$ the vertical diffusion coefficient. We suppose the earth surface $z = 0$ to be impenetrable to the diffusing pollution and to be non-absorbing, and we shall take the following boundary conditions:

$$k(z)\frac{\partial c}{\partial z} \underset{z\to 0}{\longrightarrow} 0; \qquad c \underset{z\to\infty}{\longrightarrow} 0. \tag{2}$$

We shall seek the basic solution of Equation (1), i.e. the solution of $c(z, t; h)$, corresponding to the situation when an instantaneous point source of unit intensity is present at the height $z = h$ at the moment $t = 0$: this solution satisfies the initial condition

$$c(z, t; h) \underset{t\to 0}{\longrightarrow} \delta(z - h) \tag{3}$$

where $\delta(z)$ is the delta-function.

In accordance with similarity theory for the turbulent regime in the surface layer developed by Monin and Obukhov (1953, 1954), the coefficient of turbulent diffusion has the form

$$k(z) = \kappa v_* L \cdot K\left(\frac{z}{L}\right) \tag{4}$$

where L is the length scale in the surface layer given by

$$L = \frac{v_*^3}{\kappa \dfrac{g}{T_0}\left(-\dfrac{q}{c_p \rho}\right)}, \tag{5}$$

v_* is the friction velocity, κ von Kármán constant, g the acceleration of gravity, T_0 the mean standard temperature of the surface layer, q the vertical turbulent heat flux, c_p the specific heat and ρ the air density. The dimensionless function $K(z/L)$ in this case is the same as the Richardson number. According to Monin and Obukhov when $|z/L|$ is small this function is asymptotically equal to z/L. In the case of stable stratification ($q < 0$, $L > 0$), to the consideration of which we confine ourselves in the present paper, if z/L is large the function $K(z/L)$ asymptotically approaches some constant R, which has the value of the limiting Richardson number for the stable atmosphere (this number at any rate does not exceed the critical Richardson number). Further, we shall take $R = 1$ and this will not limit the scope of the consequent results.

Let us introduce dimensionless values, putting

$$\zeta = \frac{z}{L}; \qquad \tau = \frac{\kappa v_* t}{L}; \qquad \eta = \frac{h}{L}; \qquad s = Lc. \tag{6}$$

The Equations (1)–(3) in dimensionless variables take the form

$$\frac{\partial s}{\partial \tau} = \frac{\partial}{\partial \zeta} K(\zeta) \frac{\partial s}{\partial \zeta}; \qquad K(\zeta)\frac{\partial s}{\partial \zeta} \underset{\zeta \to 0}{\longrightarrow} 0; \qquad s \underset{\zeta \to \infty}{\longrightarrow} 0; \tag{7}$$
$$s(\zeta, \tau; \eta) \underset{\tau \to 0}{\longrightarrow} \delta(\zeta - \eta).$$

The function

$$K(\zeta) = \begin{cases} \zeta & \text{if} \quad \zeta \leqslant 1 \\ 1 & \text{if} \quad \zeta \geqslant 1 \end{cases} \tag{8}$$

will be the simplest approximation of the function $K(\zeta)$ the asymptotic properties of which are indicated above. The model for the turbulent diffusion coefficient as described above was proposed by Shvetz and Yudine (1940) and was then used in a number of works. In contrast to these workers we determine the parameters of the indicated model in accordance with the similarity theory.

If $\eta < 1$ the solution of Equations (7)–(8) has the form

$$s(\zeta, \tau; \eta) = \begin{cases} \dfrac{1}{\pi}\displaystyle\int_0^\infty e^{-(\tau x^2)/4} \frac{J_0(x\sqrt{\zeta})J_0(x\sqrt{\eta})\,dx}{J_0^2(x)+J_1^2(x)} & \text{if} \quad \zeta \leqslant 1 \\ \dfrac{1}{\pi}\displaystyle\int_0^\infty e^{-(\tau x^2)/4} J_0(x\sqrt{\eta}) \frac{J_0(x)\cos\dfrac{x(\zeta-1)}{2} - J_1(x)\sin\dfrac{x(\zeta-1)}{2}}{J_0^2(x)+J_1^2(x)}\,dx & \text{if} \quad \zeta \geqslant 1; \end{cases} \tag{9}$$

whereas if $\eta > 1$ we obtain

(10)

$$s(\zeta,\tau;\eta)=\begin{cases}\dfrac{1}{\pi}\displaystyle\int_0^{\infty} e^{-(\tau x^2)/4} J_0(x\sqrt{\zeta})\,\dfrac{J_0(x)\cos\dfrac{x(\eta-1)}{2}-J_1(x)\sin\dfrac{x(\eta-1)}{2}}{J_0^2(x)+J_1^2(x)}\,dx & \text{if} \quad \zeta < 1\\[2ex] \dfrac{1}{2\sqrt{(\pi\tau)}}\left[e^{-(\zeta-\eta)^2/4\tau}-e^{-(\zeta+\eta-2)^2/4\tau}\right]+\dfrac{1}{\pi}\displaystyle\int_0^{\infty} e^{-(\tau x^2)/4} J_0(x) \\ \quad\times\dfrac{J_0(x)\cos\dfrac{x(\zeta+\eta-2)}{2}-J_1(x)\sin\dfrac{x(\zeta+\eta-2)}{2}}{J_0^2(x)+J_1^2(x)}\,dx & \text{if} \quad \zeta > 1.\end{cases}$$

The proof of Equations (9) and (10) is given in the Appendix. The values of the function $s(\zeta,\tau;0)$ for the case of a surface source of the pollution, were computed by Monin (1956); these values are given in Table 1. Fig. 1 illustrates the dependence of s upon ζ for various values of τ. Fig. 2 illustrates the dependence s upon τ when ζ takes various values.

TABLE 1.

ζ τ	0·05	0·1	0·2	0·3	0·4	0·8	1·2	2·0	3·0	4·0	6·0	8·0	10·0
0	19·999	9·999	4·999	3·333	2·500	1·258	0·847	0·540	0·398	0·325	0·252	0·213	0·188
0·1	2·707	3·679	3·033	2·388	1·947	1·111	0·784	0·518	0·386	0·320	0·250	0·212	0·187
0·2	0·366	1·353	1·839	1·711	1·517	0·981	0·726	0·496	0·377	0·314	0·247	0·210	0·186
0·3	0·049	0·498	1·116	1·226	1·118	0·869	0·673	0·478	0·368	0·309	0·245	0·209	0·185
0·4	0·007	0·183	0·677	0·879	0·921	0·770	0·624	0·460	0·360	0·304	0·242	0·207	0·184
0·5	0·001	0·067	0·410	0·630	0·718	0·684	0·580	0·443	0·352	0·300	0·240	0·206	0·183
0·6	0·000	0·025	0·249	0·452	0·561	0·609	0·539	0·426	0·344	0·295	0·238	0·204	0·182
0·7	0·000	0·009	0·151	0·324	0·436	0·543	0·502	0·411	0·337	0·290	0·235	0·203	0·181
0·8	0·000	0·003	0·092	0·232	0·344	0·485	0·469	0·396	0·329	0·286	0·233	0·202	0·180
0·9	0·000	0·001	0·056	0·166	0·271	0·434	0·436	0·382	0·322	0·282	0·231	0·200	0·179
1·0	0·000	0·000	0·034	0·119	0·215	0·390	0·410	0·368	0·316	0·277	0·229	0·199	0·178
1·1	0·000	0·000	0·021	0·085	0·170	0·350	0·383	0·355	0·308	0·273	0·226	0·197	0·177
1·2	0·000	0·000	0·013	0·061	0·132	0·312	0·356	0·342	0·301	0·268	0·224	0·196	0·176
1·3	0·000	0·000	0·007	0·045	0·102	0·276	0·330	0·328	0·293	0·264	0·222	0·194	0·175
1·4	0·000	0·000	0·004	0·031	0·078	0·242	0·304	0·318	0·285	0·259	0·219	0·192	0·173
1·5	0·000	0·000	0·003	0·021	0·058	0·212	0·279	0·306	0·277	0·255	0·216	0·190	0·172

For a steady line source of pollution with intensity C at the height $z = h$ and perpendicular to the direction of the wind x, the concentration of the pollution may be determined from the equation:

(11) $$c(x,z;h) = \frac{C}{uL}\,s\left(\frac{z}{L},\frac{xv_*x}{uL};\frac{h}{L}\right).$$

APPENDIX

Having subjected the Equation (7) to the Laplace transformation according to τ and having determined the Green function for the transformed equation, if $\eta < 1$ we obtain

$$s(\zeta,\tau;\eta) = \begin{cases} \dfrac{1}{\tau} e^{-(\zeta+\eta)/\tau} I_0\left(\dfrac{2\sqrt{(\eta\zeta)}}{\tau}\right) + \dfrac{1}{\pi i}\displaystyle\int_{a-i\infty}^{a+i\infty} e^{p\tau} I_0[2\sqrt{(p\eta)}]\, I_0[2\sqrt{(p\zeta)}] \\ \qquad \times \dfrac{K_1(2\sqrt{p})-K_0(2\sqrt{p})}{I_1(2\sqrt{p})+I_0(2\sqrt{p})}\,dp \qquad \text{if} \quad \zeta < 1 \\ \dfrac{1}{2\pi i}\displaystyle\int_{a-i\infty}^{a+i\infty} e^{p\tau-\sqrt{p}(\zeta-1)} \dfrac{I_0[2\sqrt{(p\eta)}]}{I_1(2\sqrt{p})+I_0(2\sqrt{p})}\dfrac{dp}{\sqrt{p}} \qquad \text{if} \quad \zeta > 1. \end{cases}$$

If $\eta > 1$ we obtain

$$s(\zeta,\tau;\eta) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{a-i\infty}^{a+i\infty} e^{p\tau-\sqrt{p}(\eta-1)} \dfrac{I_0[2\sqrt{(p\zeta)}]}{I_1(2\sqrt{p})+I_0(2\sqrt{p})}\dfrac{dp}{\sqrt{p}} \qquad \text{if} \quad \zeta < 1 \\ \dfrac{1}{2\sqrt{(\pi\tau)}}\left[e^{-(\zeta-\eta)^2/4\tau} - e^{-(\zeta+\eta-2)^2/4\tau}\right] \\ \qquad + \dfrac{1}{2\pi i}\displaystyle\int_{a-i\infty}^{a+i\infty} e^{p\tau-\sqrt{p}(\zeta+\eta-2)} \dfrac{I_0(2\sqrt{p})}{I_1(2\sqrt{p})+I_0(2\sqrt{p})}\dfrac{dp}{\sqrt{p}} \\ \qquad\qquad \text{if} \quad \zeta > 1. \end{cases}$$

We have already extracted the items for which the inverse Laplace transformation is finally performed. While computing the contour integrals in the above formulae, one should first of all find out the character of the singular points of the integrated functions: one of the singular points is that of branching $p = 0$. To eliminate the multi-cipheredness of the integrated function, the integration contour should be taken in a complex plane p with cut along the negative part of the real axis. In this case we choose the specimen of the complex plane for which $|\arg p| < \pi$. The zeros of the denominator $I_0(2\sqrt{p}) + I_1(2\sqrt{p})$, i.e. the roots of the function $F(z) I_0(z) + I_1(z)$ in the range of $|\arg z| < \pi/2$, could serve as other singular points: we shall prove that there are no such roots. It is obvious when $z = x > 0$ is valid, as $F(x) > 1$. Neither $F(z)$ has purely imaginary roots as $F(z) = J_0(iz) - iJ_1(iz)$ and the equality $F(iy) = 0$ is valid only when $J_0(y) = J_1(y) = 0$, but J_0 and J_1 have no common roots. Suppose that $F(z)$ has the complex root z_0: then the complex-conjugated number z_0^* will also be a root. Using the well-known formula

$$\int^{t} tJ_n(kt)J_n(lt)\,dt = \frac{t}{k^2-l^2}\left[J_n(kt)\frac{dJ_n(lt)}{dt} - J_n(lt)\frac{dJ_n(kt)}{dt}\right]$$

we obtain the relation

$$\int_0^1 t[|I_0(z_0t)|^2 + |I_1(z_0t)|^2]\,dt = \frac{I_0(z_0)\,I_1(z_0^*) + I_0(z_0^*)\,I_1(z_0)}{z_0+z_0^*} > 0.$$

If z_0 is a root of $F(z)$ then this expression can be written in the form $\frac{|I_0(z_0)|^2}{\mathrm{Re}\,z_0}$ so that $\mathrm{Re}\,z_0 < 0$ and, consequently, the function $F(z)$ has no complex roots in the range $|\arg z| < \pi/2$; so the point of branching $p = 0$ is the only singular point of the integrated functions in our equations. In this case the integration contour can be reduced to the edges of cut and after an appropriate transformation of the contour integrals the Equations (9) and (10) can be obtained.

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